Theory Thévenin Equivalent

Approach #1
1. Calculate $V_{oc}$ and $I_{sc}$
2. Shorten a, b, the current between a and b
3. Voltage between a and b when a, b is open

Approach #2
1. Calculate either $V_{oc}$ or $I_{sc}$
2. Calculate $R_{th}$ directly
Norton’s Theorem

Linear two-terminal circuit can be replaced by an equivalent circuit composed of a current source and parallel resistor

\[ i_N = \frac{v_{Th}}{R_{Th}} \]

Current through output with short circuit

\[ R_N = R_{Th} \]

Resistance at terminals with all circuit sources set to zero

Thévenin and Norton Equivalency

Power Transfer

In many situations, we want to maximize power transfer to the load

\[ P_L = i_Lv_L = \frac{v_s^2 R_L}{(R_s + R_L)^2} \]

\[ P_L = i_L^2 V_L, \quad I_L = \frac{V_L}{R_s + R_L} \]

Maximum power when \( R_L = R_s \)

\[ P_{L(max)} = \frac{v_s^2 R_L}{(R_s + R_L)^2} = \frac{v_s^2}{4R_s} \]

Load Resistance = \( R_{th} \)

\[ R_L = R_{th} \]

\[ P_{max} = \frac{V_s^2}{4R_{th}} \]
**Thevenin Equivalent Circuit**

![Thevenin Circuit Diagram]

Approach #1 both $V_{oc}$ and $I_{sc}$
- **open circuit** $V_{th} = V_{oc}$
- **short circuit** $R_{th} = \frac{V_{oc}}{I_{sc}}$

Approach #2
- **step 1** either $V_{oc}$ or $I_{sc}$
- **step 2** $R_{th}$ (shorten the voltage source, open the current source)

**Norton Equivalent Circuit**

- **Step 1** Thevenin Analysis
- **Step 2** $V_{th} = \frac{V_{th}}{R_{th}} \rightarrow I \left( \frac{1}{R_{th}} \right)$

**Maximum Power Transfer**

- **Step 1** Thevenin Analysis
\[ R_L = R_{th} \]

\[ P_{RL} = \left( \frac{V_{th}}{2R_{th}} \right)^2 \cdot R_L = \frac{V_{th}^2}{4R_{th}} \]
Exercise 3-14  The bridge circuit of Fig. E3-14 is connected to a load $R_L$ between terminals $(a, b)$. Choose $R_L$ such that maximum power is delivered to $R_L$. If $R = 3$ Ω, how much power is delivered to $R_L$?

$$P_L = \frac{V^2}{R_L}$$

Solution: We need to remove $R_L$ and then determine the Thévenin equivalent circuit at terminals $(a, b)$.

Open-circuit voltage:

The two branches are balanced (contain same total resistance of $3R$). Hence, identical currents will flow, namely

$$I_1 = I_2 = \frac{24}{3R} = \frac{8}{R}.$$  

$$V_{oc} = V_a - V_b = 2RI_1 - RI_2 = RI_1 = R \frac{8}{R} = 8 \text{ V}.$$  

To find $R_{TB}$, we replace the source with a short circuit:

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Hence,

and the Thévenin circuit is

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For maximum power transfer with \( R = 3 \ \Omega \), \( R_L \) should be

\[
R_L = \frac{4R}{3} = \frac{4 \times 3}{3} = 4 \ \Omega,
\]

and

\[
P_{\text{max}} = \frac{v^2}{4R_L} = \frac{8^2}{4 \times 4} = 4 \ \text{W}.
\]
Homework
Tuesday, July 23, 2019  1:37 PM

Problem 3.81  What value of the load resistor $R_L$ will extract the maximum amount of power from the circuit in Fig. P3.81, and how much power will that be?

\[
\begin{align*}
\{ & \frac{V_a}{3} + \frac{V_a - V_b}{4} = 0 \Rightarrow 5V_a - 3V_b = 36 \\
& \frac{V_b - V_c}{3} + \frac{V_c + V_b}{6} = 0 \Rightarrow 6V_b - 6V_c + 3V_c + 4V_b = 0 \Rightarrow 13V_b = 6V_c \\
& \Rightarrow 5\left(\frac{V_b}{2} - V_c\right) - 3V_c = 3b \Rightarrow 65\frac{V_b}{2} - 18V_c = 21b \\
& \Rightarrow V_c = \frac{36}{27}V \Rightarrow \frac{V_c}{2} = \frac{8}{9}V \\
\} \\
\text{Problem 3.81: Circuit for Problem 3.81.}
\end{align*}
\]

Solution: We start by obtaining the Thévenin equivalent circuit at terminals $(a,b)$, as if $R_L$ were not there. We first find $V_{oc}$:

\[
\begin{align*}
\frac{V}{6} - 3 + \frac{V}{12} = 0 \\
V = 12V.
\end{align*}
\]

Hence,

\[
\begin{align*}
\text{Voltage division gives:} \\
V_{Th} = V_{oc} = \left(\frac{8}{4+8}\right)V = \frac{8}{12} \times 12 = 8V.
\end{align*}
\]

Next, we suppress the current source to find $R_{Th}$:

\[
\begin{align*}
R_{Th} = 10.44 \Omega.
\end{align*}
\]

Equivalent circuit:

\[
\begin{align*}
\text{For maximum power transfer to } R_L, \\
R_L = R_{Th} = 10.44 \Omega \\
I = \frac{8}{2 \times 10.44} = 0.38 \text{ A} \\
P_{\text{max}} = I^2 R_L = (0.38)^2 \times 10.44 = 1.53 \text{ W}.
\end{align*}
\]

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**Problem 3.82** For the circuit in Fig. P3.82, choose the value of $R_L$ so that the power dissipated in it is a maximum.

\[
R_{\text{th}} = \frac{2 + 6}{1/(4 + 8)}
\]

\[
= \frac{8}{8/12} = 4.8 \, \text{k}\Omega
\]

**Solution:** We need to find the Thévenin equivalent circuit at terminals $(a, b)$, as if $R_L$ were not present.

The current source will divide among $I_1$ and $I_2$ such that

\[
(4 + 2)I_1 = (8 + 6)I_2 \Rightarrow \frac{I_1}{I_2} = \frac{14}{6} = \frac{7}{3}
\]

Also, $I_1 + I_2 = 2$ mA

The solution yields:

\[
I_1 = 1.4 \, \text{mA}, \quad I_2 = 0.6 \, \text{mA},
\]

\[
V_{oc} = (-4I_1 + 8I_2) \times 10^3
\]

\[
= -4 \times 1.4 + 8 \times 0.6 = -0.8 \, \text{V}.
\]

To find $R_{\text{th}}$, we suppress the current source and simplify the circuit:

\[
R_{\text{th}} = 8 \, \text{k}\Omega \parallel 12 \, \text{k}\Omega = 4.8 \, \text{k}\Omega
\]

Hence, $R_L$ should be 4.8 kΩ for maximum power transfer to it.

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# Summary

## Chapter 3 Relationships

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<td>$\sum \text{of all current leaving a node} = 0$</td>
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<td>$\sum \text{of all voltages around a loop} = 0$</td>
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<td>$R_L = R_s$</td>
<td>$P_L(\text{max}) = \frac{v_s^2}{4R_L}$</td>
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</table>
Problem 3.4 For the circuit in Fig. P3.4:

(a) Apply nodal analysis to find node voltages $V_1$ and $V_2$.

(b) Determine the voltage $V_R$ and current $I$.

\[
\begin{align*}
\frac{V_1 - 16}{1} + \frac{V_2}{\frac{1}{6}} + \frac{V_2 - V_3}{\frac{1}{8}} &= 0 \\
\frac{V_3 - 10}{\frac{1}{6}} + \frac{V_2}{\frac{1}{6}} + \frac{V_3}{\frac{1}{8}} &= 0 \\
\end{align*}
\]

Figure P3.4: Circuit for Problem 3.4.

Solution: (a) At nodes $V_1$ and $V_2$,

Node 1: \[\frac{V_1 - 16}{1} + \frac{V_2}{1} + \frac{V_1 - V_2}{1} = 0 \quad \checkmark \quad (1)\]

Node 2: \[\frac{V_2 - V_1}{1} + \frac{V_2}{1} + \frac{V_2}{1} = 0 \quad \checkmark \quad (2)\]

Simplifying Eqs. (1) and (2) gives:

\[
\begin{align*}
3V_1 - V_2 &= 16 \quad (3) \\
-V_1 + 3V_2 &= 0 \quad (4)
\end{align*}
\]

Simultaneous solution of Eqs. (3) and (4) leads to:

\[V_1 = 6 \text{ V}, \quad V_2 = 2 \text{ V}.\]

(b)

\[V_R = V_1 - V_2 = 6 - 2 = 4 \text{ V} \quad \checkmark \]

\[I = \frac{V_2}{1} = \frac{2}{1} = 2 \text{ A} \quad \checkmark \]
Problem 3.3  Use nodal analysis to determine the current $I_x$ and amount of power supplied by the voltage source in the circuit of Fig. P3.3.

![Circuit Diagram](image)

Figure P3.3: Circuit for Problem 3.3.

Solution: At node $V$, application of KCL gives

$$-9 + \frac{V}{2} + \frac{V}{4} + \frac{V-40}{8} = 0$$

$$V \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right) = 9 + \frac{40}{8}$$

$$\frac{7V}{8} = 9 + 5$$

$$V = 16 \text{ V.}$$

The current $I_x$ is then given by

$$I_x = \frac{V}{4} = \frac{16}{4} = 4 \text{ A.}$$

To find the power supplied by the 40-V source, we need to first find the current $I$ flowing into its positive terminal,

$$I = \frac{V - 40}{8} = \frac{16 - 40}{8} = -3 \text{ A.}$$

Hence,

$$P = VI = 40 \times (-3) = -120 \text{ W}$$

(The minus sign confirms that the voltage source is a supplier of power.)
**Problem 3.26** Apply mesh analysis to find the mesh currents in the circuit of Fig. P3.26. Use the information to determine the voltage $V$.

\[
\begin{align*}
-16 + 2I_1 + 3(I_2 - I_1) &= 0 \\
3(I_1 - I_2) + 2I_2 + 4I_1 &= 0
\end{align*}
\]

**Solution:** Application of KVL to the two loops gives:

Mesh 1: \[-16 + 2I_1 + 3(I_1 - I_2) = 0,\]
Mesh 2: \[3(I_1 - I_2) + (2 + 4)I_2 = 0,\]

which can be simplified to

\[
\begin{align*}
5I_1 - 3I_2 &= 16 \\
-3I_1 + 9I_2 &= -12.
\end{align*}
\]

Simultaneous solution of (1) and (2) leads to

\[
\begin{align*}
I_1 &= 3 \text{ A}, \\
I_2 &= -\frac{1}{3} \text{ A}.
\end{align*}
\]

Hence,

\[
V = 3(I_1 - I_2) = 3 \left( 3 - \frac{1}{3} \right) = 10 \text{ V}.
\]

**Approach #2**

\[
V = 16 - 2I_1 = 16 - 2 \times 3 = 10 \text{ V}
\]

**Approach #3**

\[
V = 12 + 2I_2 + 4I_1 = 12 + 2 \times (-\frac{1}{3}) + 4 \times (-\frac{1}{3}) = 12 - 2 = 10 \text{ V}
\]

**Approach #1**

\[
V = 3(I_1 - I_2) = 3 \times \left( 3 - \frac{1}{3} \right) = 10 \text{ V}
\]

\[
\frac{V - 16}{2} + \frac{V}{3} + \frac{V - 12}{2 + 4} = 0 \\
3(V - 16) + 2V + V - 12 = 0 \\
6V - 48 - 12 = 0 \\
6V = 60 \\
V = 10 \text{ V}
\]
**Problem 3.43** Apply mesh analysis to the circuit of Fig. P3.43 to find $I_x$.

![Circuit Diagram](image)

**Figure P3.43** Circuit for Problem 3.43.

**Solution:**

Mesh 1: \[-4 + I_1 + 0.1(I_1 - I_2) + 0.2(I_1 - I_3) = 0\]
Mesh 2: \[0.1(I_2 - I_1) + 0.2I_2 + (I_2 - I_3) = 0\]
Mesh 3: \[0.2(I_3 - I_1) + (I_3 - I_2) + 0.1I_3 = 0\]

\[
\begin{align*}
I_1 &= 3.48 \text{ A}, \quad I_2 = 1.67 \text{ A}, \quad I_3 = 1.82 \text{ A} \\
I_x &= I_3 - I_2 = 1.82 - 1.67 = 0.15 \text{ A}.
\end{align*}
\]
Problem 3.53 Use the by-inspection method to establish a node-voltage matrix equation for the circuit in Fig. P3.53. Solve the matrix equation by MATLAB® or MathScript software to find $V_1$ to $V_4$.

![Circuit diagram](image)

Figure P3.53: Circuit for Problem 3.53.

Solution:

$$G_{11} = \frac{1}{2+1} + \frac{1}{3+4} = 0.476$$

$$G_{12} = G_{21} = \frac{1}{2+1} = -0.333$$

$$G_{13} = G_{31} = 0$$

$$G_{14} = G_{41} = -\frac{1}{3+4} = -0.143$$

$$G_{22} = \frac{1}{1+2} + \frac{1}{7} + \frac{1}{6} = 0.643$$

$$G_{23} = G_{32} = -\frac{1}{6} = -0.167$$

$$G_{24} = G_{42} = 0$$

$$G_{33} = \frac{1}{5} + \frac{1}{6} + \frac{1}{9} = 0.478$$

$$G_{34} = G_{43} = -\frac{1}{5} = -0.2$$

$$G_{44} = \frac{1}{3+4} + \frac{1}{5} = 0.343$$

Application of Eq. (3.26) gives:

$$\begin{bmatrix}
0.476 & -0.333 & 0 & -0.143 \\
-0.333 & 0.643 & -0.167 & 0 \\
0 & -0.167 & 0.478 & -0.2 \\
-0.143 & 0 & -0.2 & 0.343
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
0 \\
-2 \\
-3
\end{bmatrix}$$

Matrix inversion gives:

$$V_1 = -8.1689 \text{ V}, \quad V_2 = -8.4235 \text{ V}, \quad V_3 = -16.155 \text{ V}, \quad V_4 = -21.5748 \text{ V}.$$
Problem 3.55  Find $I_0$ in the circuit of Fig. P3.55 by developing a mesh-current matrix equation and then solving it using MATLAB® or MathScript software.

![Circuit Diagram](image)

**Figure P3.55** Circuit for Problem 3.55.

Solution: Application of Eq. (3.29) gives:

$$
\begin{bmatrix}
50 & -20 & -20 & 0 \\
-20 & 80 & -20 & 0 \\
-20 & -20 & 50 & -10 \\
0 & 0 & -10 & 50
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
0 \\
0 \\
0
\end{bmatrix}
$$

Matrix inversion yields:

$$
I_1 = 0.54 \text{ A}, \quad I_2 = 0.36 \text{ A}, \quad I_3 = 0.38 \text{ A}, \quad I_4 = 0.08 \text{ A} \\
I_0 = I_3 - I_4 = 0.37 - 0.07 = 0.3 \text{ A}.
$$
Exercise 3-11  Determine the Thévenin-equivalent circuit at terminals \((a,b)\) in Fig. E3-11.

\[ R_{th} = \frac{(2 + 3)}{115} = \frac{5}{115} = 0.5 \Omega \]

**Figure E3.11**

**Solution:**

(1) **Open-circuit voltage**

We apply node voltage method to determine open-circuit voltage:

\[ \frac{V_1}{2} - 4 + \frac{V_1 - V_2}{3} = 0, \]

\[ \frac{V_2 - V_1}{3} + 3 + \frac{V_2}{5} = 0. \]

Solution gives: \(V_2 = -3.5\) V.

Hence, \(V_{th} = V_{oc} = -3.5\) V.

(2) **Short-circuit current**

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Because of the short circuit,

\[ V_2 = 0. \]

Hence at node \( V_i \):

\[
\begin{align*}
\frac{V_i}{2} - 4 + \frac{V_i}{3} &= 0 \\
V_i \left( \frac{1}{2} + \frac{1}{3} \right) &= 4 \\
V_i &= \frac{24}{5} \text{ V} \\
l_1 &= \frac{V_i}{3} = \frac{24}{5 \times 3} = \frac{8}{5} \text{ A}, \\
l_{sc} &= l_1 - 3 - \frac{8}{5} - 3 = -\frac{7}{5} = -1.4 \text{ A} \\
R_{th} &= \frac{V_{th}}{l_{sc}} = \frac{-3.5}{-1.4} = 2.5 \text{ } \Omega.
\end{align*}
\]

Thévenin equivalent:

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Problem 3.65  Find the Thévenin equivalent circuit at terminals \((a, b)\) for the circuit in Fig. P3.65.

\[
\begin{align*}
V &= \frac{V}{4} + \frac{V}{6} + \frac{V - 2}{3} = 4 \\
\text{Hence, } V &= \frac{56}{9} \text{ V.} \\
V_{\text{Th}} = V_{\text{oc}} &= V - 2 = \frac{56}{9} - 2 = 4.22 \text{ V.}
\end{align*}
\]

Suppressing the sources:

\[
R_{\text{Th}} = \frac{4}{3} + 2.5 = 3.83 \text{ }\Omega
\]

Thévenin equivalent circuit:
Because of the short circuit, \( V_2 = 0 \).

Hence at node \( V_1 \):

\[
\frac{V_1}{2} - 4 + \frac{V_1}{3} = 0
\]

\[
V_1 \left( \frac{1}{2} + \frac{1}{3} \right) = 4
\]

\[
V_1 = \frac{24}{5} \text{ V}
\]

\[
I_1 = \frac{V_1}{3} = \frac{24}{5 \times 3} = \frac{8}{5} \text{ A},
\]

\[
I_{ce} = I_1 - 3 = \frac{8}{5} - 3 = -\frac{7}{5} = -1.4 \text{ A}
\]

\[
R_{Th} = \frac{V_{th}}{I_{ce}} = \frac{-3.5}{-1.4} = 2.5 \text{ } \Omega
\]

Thévenin equivalent:

![Thévenin equivalent circuit diagram]

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Exercise 3-14  The bridge circuit of Fig. E3-14 is connected to a load $R_L$ between terminals $(a,b)$. Choose $R_L$ such that maximum power is delivered to $R_L$. If $R = 3 \, \Omega$, how much power is delivered to $R_L$?

\[ 24 \, V \]

![Figure E3-14](image)

\textbf{Solution:} We need to remove $R_L$ and then determine the Thévenin equivalent circuit at terminals $(a,b)$. Open-circuit voltage:

\[ 24 \, V \]

The two branches are balanced (contain same total resistance of $3R$). Hence, identical currents will flow, namely

\[ I_1 = I_2 = \frac{24}{3R} = \frac{8}{R} \, \text{A}. \]

\[ V_{oc} = V_a - V_b = 2RI_1 - RI_2 = RI_1 = \frac{8}{R} R = 8 \, \text{V}. \]

To find $R_{TB}$, we replace the source with a short circuit:

---

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\[ R \parallel 2R = \frac{R \times 2R}{R + 2R} = \frac{2}{3} R \]

Hence,

\[ R_{\text{Th}} = \frac{4R}{3} \]

and the Thévenin circuit is

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For maximum power transfer with $R = 3 \, \Omega$, $R_L$ should be

$$R_L = \frac{4R}{3} = \frac{4 \times 3}{3} = 4 \, \Omega,$$

and

$$P_{\text{max}} = \frac{V_s^2}{4R_L} = \frac{8^2}{4 \times 4} = 4 \, \text{W}.$$
Problem 3.81  What value of the load resistor $R_L$ will extract the maximum amount of power from the circuit in Fig. P3.81, and how much power will that be?

![Figure P3.81: Circuit for Problem 3.81.](image)

**Solution:** We start by obtaining the Thévenin equivalent circuit at terminals $(a,b)$, as if $R_L$ were not there. We first find $V_{\text{th}}$:

\[
\begin{align*}
\frac{V}{6} - 3 + \frac{V}{12} &= 0 \\
V &= 12 \text{ V}.
\end{align*}
\]

Hence,

\[
\frac{V}{6} = 2 \text{ V}.
\]

Voltage division gives:

\[
V_{\text{th}} = V_{\text{oc}} = \left(\frac{8}{4+8}\right) V = \frac{8}{12} \times 12 = 8 \text{ V}.
\]

Next, we suppress the current source to find $R_{\text{th}}$:

\[
R_{\text{th}} = 10.44 \Omega.
\]

Equivalent circuit:

![Equivalent circuit](image)

For maximum power transfer to $R_L$,

\[
R_L = R_{\text{th}} = 10.44 \Omega
\]

\[
I = \frac{8}{2 \times 10.44} = 0.38 \text{ A}
\]

\[
P_{\text{max}} = I^2 R_L = (0.38)^2 \times 10.44 = 1.53 \text{ W}.
\]