Two-Way Communication Source Coding

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Abstract

In this paper we will discuss two-way communication. The topic will cover source coding for two sources communicating over a two-way channel, and some of the techniques used with relay channels. We will explore the results in the papers titled Intercitive Data Compression [1], Two-way Source coding with a Fidelity Criterion [2], Two-way Source Coding with a Helper [4], Two-way source coding with a common Helper [3], Two-way source coding through a Relay [5], and An Achievable Rate Region for the two-way two-relay channel [6].

(Note: None of this is my work, it is just a summary of what was done by the authors of different papers in the reference section.)

I. INTRODUCTION

Two-way communication is somewhat a young topic in information theory. The different references used in this paper will highlight some of the different techniques used to address some of the problems in two-way communication. Most of the papers addressed the unknow about this topic by setting bounds on code lengths and studying the achievability of the codes generated. What was noticed in most of the papers is that the correlation between the two sources was important. Whether correlation was directly applied by having the two sources have joint distribution, or induced by adding a helper or a relay channel to the configuration, correlation was important. Let's introduce some of the major topics, their theory, and their results without going into much detail about their proofs.

II. A GENERAL APPROACH

A. Data Compression Set up

El Gamal and Orlitsky approached this by starting off with Shannon's general approach to the general one-way channel theory. In general they assumed communication over a noiseless two-way channel and modeled the channel to be generalized discrete time binary channel [1]. Here is the framework that was used: Let X, Y be two random variables distributed over a finite set \( X \times Y \) with joint intropy \( H(X, Y) \) and marginal entropies \( H(X) \) and \( H(Y) \) and \( H(X|Y) = H(X, Y) - H(Y) \) and \( H(Y|X) = H(X, Y) - H(X) \) [1]. It was also assumed that source X knows and source Y knows y, and they defined a set of sequences A and B with cardinality in consideration. They setup a prefix code where this code is a subset of \( X \times Y \). The prefix code ensures that the message is uniquely decodable and make the message deterministic [1]. They proved most of their work by setting up upper and lower bounds based on the properties of prefix and huffman codes.

1- Atleast \( H(X|Y) + H(Y|X) \) bits must be exchanged and \( H(X, Y) + 2 \) bits are sufficient.
2- For \( p(x, y) > 0 \) for all \( (x, y) \), then atleast \( H(X, Y) \) bits must be communicated on average. 3- If \( p(x, y) \) is uniform over its support set then the average number of bits needed is close to \( H(X|Y) + H(Y|X) \) [1].

B. Theorems and Lemmas

Lemma 1. [1] Let \( C \) be a code for \( S \). For every \( (x, y), (x', y') \in S \), if \( (x, y') \in S \) and one of \( B^{XY}(x, y) \), \( B^{XY}(x', y') \) is a prefix of the other then, \( B^{XY}(x, y) = B^{XY}(x, y') = B^{XY}(x', y') \).

Corollary 1. [1] Let \( C \) be an exchange code for \( S \). if \( (x, y), (x', y') \) are distinct members of \( S \) and \( (x, y') \in S \), then neither one of \( B^{XY}(x, y), B^{XY}(x', y') \) is a prefix of the other (nor can they be equal).

Corollary 2. [1] If \( C \) is an exchange code for \( S \) then for every \( x \in X \), \( B^{Y}(x, y) : (x', y') \in S \) is prefix free with cardinality \( |y : (x, y) \in S| \) and for every \( y \in Y \), \( B^{X}(x, y) : (x, y) \in S \) is prefix free with cardinality \( |x : (x, y) \in S| \).

Theorem 1. [1] \( H(X|Y) + H(Y|X) \leq L_a(p) \leq H(X, Y) + 2. \)
Theorem 2. [1] If for all \((x, y) \in X \times Y, p(x, y) > 0\) then, \(H(X, Y) \leq L_a(p) \leq H(X, Y) + 2\).

Lemma 2. [1] If for all \(x \in X, |y : p(x, y) > 0| \leq n\) and for all \(y \in Y, |x : p(x, y) > 0| \leq m\), then \(L_m(p) \leq \lfloor \log(m.n) \rfloor + \lfloor \log(min(m,n)) \rfloor\).

Lemma 3. [1] Let \(V\) be a set of even size \(v > X\) discrete random variables taking values in \(X\) and \(Y\), which are finite set[2]. Now in Fig.1 \(L_n\) for all \(V\) \(\epsilon > X\).

Then, given \(L_m(p)\) for all \(x \in X\).

Theorem 3. [1] Let \(V\) be a set of size \(v\) and for \(i \in [1, ..., e]\), \(E_i \subseteq V\) and \(|E_i| \leq m\).

Then, given \(\epsilon > 0\), there exists \(C(\epsilon)\) such that for all \(p \geq (\ln \sqrt{ev})^{1+\epsilon}, p > 1\) it is possible to find a partition \(V_1, ..., V_{\lceil C(\epsilon), m/p \rceil}\).

Theorem 3. [1] Given \(\epsilon > 0\), there exists \(C(\epsilon)\) such that for all probability distribution \(p\) satisfying \(|y : p(x, y) > 0| \leq n\) for all \(x \in X\) and \(|x : p(x, y) > 0| \leq m\) for all \(y \in Y\), \(L_m(p) \leq \log(m.n) + (1 + \epsilon).\log\log(max(|X|, |Y|)) + C(\epsilon)\).

Theorem 5. [1] For any \(\epsilon > 0\) there exists \(C(\epsilon)\) such that if \(p\) is equiprobable then, \(L_a(p) \leq H(X|Y) + H(Y|X) + (3 + \epsilon)\log\log(max(|X|, |Y|)) + C(\epsilon)\).

C. Results

1- Atleast \(H(X|Y) + H(Y|X)\) bits must be exchanged and \(H(X, Y) + 2\) bits are sufficient.
2- For \(p(x, y) > 0\) for all \((x, y)\), then at least \(H(X, Y)\) bits must be communicated on average.
3- If \(p(x, y)\) is uniform over its support set then the average number of bits needed is close to \(H(X|Y) + H(Y|X)[1]\).

III. USING FIDELITY CRITERION

Now lets take a more specific approach to the two-way communication channel using fidelity criterion or in other words distortion. The model is depicted in figure 1.

![Fig. 1. Two-way source coding scheme.](image)

In this paper Kaspi, started with this framework, for a sequence \([X_i, Y_i]_{i=1}^{\infty}\) generated from the pair \((X, Y)\) discrete random variables taking values in \(X\) and \(Y\), which are finite set[2]. Now in Fig.1 \(X_i\) which is part of \(X_i, Y_i\) at X side will be transmitted to Y and observed by it as X, and vice versa for Y as the transmitter[2]. Block coding was used in this paper where an n-vector \(x = (x_1, x_2, ..., x_n)\) and an n-vector \(y = (y_1, y_2, ..., y_n)\) are received at the corresponding side[2]. Here is how it will work X will send \(nR_y^1\) bits to Y, then Y will send \(nR_x^1\) bits as a function of Y and the received bits[2]. Then X will send \(nR_x^2\) bits to Y as a function of
X and the bits received and so forth[2]. The Rates are:
\[ R_x = \sum_{k=1}^{\infty} R_x^k \text{ bits per symbol, where } k \text{ denotes the number of transmissions}[2]. \]
\[ R_y = \sum_{k=1}^{\infty} R_y^k \text{ bits per symbol, where } k \text{ denotes the number of transmissions}[2]. \]

A. The more formal definition of the problem
Kaspi, definds the problem as follows using the above as a framework. Now for the system in Fig. 1 the system is defined by \( X, Y, \hat{X}, \hat{Y}, [x_i, y_i]_{i=1}^{\infty} \) and a distortion measure \( \rho_x(x, \hat{x}), \rho_y(y, \hat{y}) \). Then he defined \( (n, k, [f^k]_{k=1}^n, [g^k]_{k=1}^n, F, G, D_x, D_y) \) to be a k-step information exchange scheme, where \( f^k \) and \( g^k \) are the encoding functions and \( F \) and \( G \) be the decoding functions and \( D_x, D_y \) are the distortion functions.

B. Results
\( J^n = \mathcal{F}^k \), where \( \mathcal{J} \) is the subset of acheivable rates and distortions[2].

IV. USING A COMMON HELPER
In this section, the idea is very similar to the fidelity criterion, but it modifies it using a helper. See Fig. 2

\[ \begin{array}{c}
\hat{Z} \\
X \\
R_3 \\
User X \\
R_2 \\
User Z \\
R_1 \\
Helper \\
Y \\
\end{array} \]

Fig. 2. Two-way source coding with a common helper[3][4].

A. Setup
Here is how Permuter, Steinberg, and Weissmen setup the problem:
-Helper sends to \( Z \) and \( X \) at \( R_1[3][4] \).
-then \( Z \) send to \( X \) at \( R_2[3][4] \).
-then \( X \) sends to \( Z \) at \( R_3[3][4] \).
-Z will send after receiving one message, \( X \) will send after receiving two messages[3][4]. All sequences are assumed to be i.i.d Markov chain \( Y - Z - X[3][4] \). Now this is how it will work, \( X \) will get two messages one from the helper and one from \( Z \) and then \( X \) will reconstruct \( X^n \) with distortion \( E[\frac{1}{n} \sum_{i=1}^{n} d_x(X_i, \hat{X}_i)] \leq D_x[3][4] \). And the same for \( Z \), where \( Z \) will get two messages one from the helper and one from \( X \) and then \( Z \) will reconstruct \( X^n \) with distortion \( E[\frac{1}{n} \sum_{i=1}^{n} d_x(X_i, \hat{X}_i)] \leq D_x[3][4] \).

B. Theorems and Lemmas
Theorem 1. [4] In the two-way rate distorion problem with a helper, as depicted in Fig. 2, where \( Y - X - Z \) \( R^O(D_x, D_z) = R(D_x, D_z) \) where the region \( R(D_x, D_z) \) is specified below.
Lemma 2. [4]:
1) The region \( R(D_x, D_z) \) is convex. 2) To exhaust \( R(D_x, D_z) \), it is enough to restrict the alphabet of \( U, V, \) and \( W \) to satisfy
|U| ≤ |Y| + 3
|V| ≤ |Z||U| + 3
|W| ≤ |U||V| |X| + 1

C. Special Cases

Fig. 3. Wyner-Ziv problem with a helper[3][4].

In the previous case we assume the Markov form $Y - X - Z$ and in the second case we assume a Markov form $Y - Z - X$ [4].

Fig. 4. Two-way multistage with a helper[3][4].

"Here we consider the two-way multistage rate-distortion problem with a helper. First, the helper sends a common message to both users, and then users $X$ and $Z$ send to each other a total rate $R_x$ and $R_z$, respectively, in K rounds" [4].

"Gaussian instance of the two way setting with a helper as dened in Section IV (A) and explicitly express the region for a mean square error distortion." [4]
Fig. 5. Gaussian two-way with a helper[3][4].

Fig. 6. Wyner-Ziv problem with a helper where the Markov chain $Y \rightarrow X \rightarrow Z$ holds[3][4].

In this Fig. "we investigate two properties of the rate-region of the Wyner-Ziv setting with a Markov form $Y \rightarrow X \rightarrow Z."[4]

D. Results

The result is the achievable region $R(D_x, D_z)$ which is defined as the set of all rate triple $(R_1, R_2, R_3)[3][4]$, where,

$$R_1 \geq I(Y; U|Z) \ [3][4]$$
$$R_2 \geq I(Z; V|U, X) \ [3][4]$$
$$R_3 \geq I(X; W|U, V, Z) \ [3][4]$$

for $p(x, y, z, u, v, w) = p(x, y)p(z|x)p(w|y)p(v|u, z)p(w|u, v, x)$, where $U, V, W$ are the auxiliary random variable with bounded cardinality[3][4].

The rewritten distortion form is:

$$Ed_x(X, \hat{X}(U, W, Z)) \leq D_x \ [3][4]$$
$$Ed_z(Z, \hat{Z}(U, V, X)) \leq D_z \ [3][4]$$

The main result of all this is the operational achievable region:

$$R^O(D_x, D_z) = R(D_x, D_z)$$

where we have the Markov chain $Y \rightarrow X \rightarrow Z$ and $R(D_x, D_z)$ as defined above. [3][4]

E. Key notes

$R(D_x, D_z)$ is convex[4].

And $U, V, W$ are restricted by the following:

$|U| \leq |Y| + 4 \ [4]$  
$|V| \leq |Z||U| + 3 \ [4]$  
$|W| \leq |U||V||X| + 1 \ [4]$
V. USING A RELAY

Fig. 7. Two-way source coding through a relay [5].

A. Formulation of the problem

According to Han and El Gamal, "A \((2^{nR}, 2^{nR_2}, 2^{nR_3}, n)\) code for the two-way source coding through a relay problem with 2-discrete memoryless source \((X_1, X_2)\) and distortion measures \(d_1\) and \(d_2\) consists of [5]:

1) 2 source encoders Encoder \(j = 1, 2\) assigns \(m_j(x^n_j) \in [1 : 2^{nR_j}]\) to each source sequence \(x^n_j \in \mathcal{X}^n_j\) [5].

2) A relay encoder that assigns an index \(m_3(m_1, m_2) \in [1 : 2^{nR_3}]\) to each index pair \((m_1, m_2) \in [1 : 2^{nR_2}] \times [1 : 2^{nR_2}]\) [5].

3) Two decoders Decoder \(j = 1, 2\) assigns an estimate \(\hat{x}^{n,j}_3(m_3, x^n_j) \in \mathcal{X}^n_3\) to each pair \((m_3, x^n_j) \in [1 : 2^{nR_3} \times \mathcal{X}^n_j]\) [5].

B. Theorems

Theorem 1. [5] Any achievable rate triple \((R_1, R_2, R_3)\) for distortion pair \((D_1, D_2)\) must satisfy the conditions

\[
R_1 \geq R_{W1}^{WZ}(D_1)
\]

\[
R_2 \geq R_{W2}^{W1}(D_2)
\]

\[
R_3 \geq \max[R_{1|2}(D_1), R_{2|1}(D_2)]
\]

Theorem 2. [5] The compress-linear code inner bound on the rate distortion region \(\mathcal{R}(D_1, D_2)\) consists of the set of rate triples \((R_1, R_2, R_3)\) such that

\[
R_1 \geq R_{W1}^{WZ}(D_1)
\]

\[
R_2 \geq R_{W2}^{W1}(D_2)
\]

\[
R_3 \geq \max[R_{1|2}^{WZ}(D_1), R_{2|1}^{WZ}(D_2)]
\]

Theorem 3. [5] The compute-compress inner bound on the rate distortion \(\mathcal{R}(D_1, D_2)\) consists of the set of rate triples \((R_1, R_2, R_3)\) such that

\[
R_1 \geq I(X_1; U_1 | X_2, Q)
\]

\[
R_2 \geq I(X_2; U_2 | X_1, Q)
\]

\[
R_1 + R_2 \geq I(X_1, X_2; U_1, U_2 | Q)
\]

\[
R_3 \geq I(V; W | X_1, Q)
\]

\[
R_3 \geq I(V; W | X_2, Q)
\]

for some \(p(q)p(u_1|x_1, q)p(u_2|x_2, q)p(w|v, q)\) and functions \(v(x_1, x_2), \hat{x}_2(w, x_1)\) such that \(H(V(X_1, X_2)|U_1, U_2) = 0\) and

\[
E(d_1(X_1, \hat{X}_1(W, X_2))) \leq D_1,
\]

\[
E(d_2(X_2, \hat{X}_2(W, X_1))) \leq D_2.
\]

Theorem 4. [5] Any rate triple \((R_1, R_2, R_3)\) achievable with distortion pair \((D_1, D_2)\) must satisfy

\[
R_1 \geq I(X_1; U_1 | X_2),
\]

\[
R_2 \geq I(X_2; U_2 | X_1),
\]

\[
R_3 \geq I(X_1; V | X_2, U_2),
\]

\[
R_3 \geq I(X_2; V | X_1, U_1),
\]

for some \(p(u_1, u_2|x_1, x_2)p(v|u_1, u_2), \hat{x}_1(v, x_2, u_2), \) and \(\hat{x}_2(v, x_1, u_1)\) such that \(U_1 \rightarrow X_1 \rightarrow X_2 \rightarrow X_1, V \rightarrow (X_1, U_2) \rightarrow X_2, V \rightarrow (X_2, U_1) \rightarrow X_1, \) and

\[
E(d_1(X_1, \hat{X}_1(W, X_2))) \leq D_1,
\]

\[
E(d_2(X_2, \hat{X}_2(W, X_1))) \leq D_2.
\]
C. Results

The result of this research only proved reliable in some special cases where the inner bound, cut-set bound and compute-compress inner bound have different outcomes. When the cut-set bound and the compute-compress inner bound coincide the relay communication rate is higher. When only the compute-compress inner bound is met, it will achieve a lower relay communication but it will increase the source rate beyond the cut-set bound [5].

VI. TWO-WAY TWO-WAY RELAY CHANNEL

A. General Idea

According to Xie and Ponniah, the idea is that "the relay linearly combines (or XOR’s) the two source messages that it decoded and then broadcasts the result, then the source nodes knowing what they transmitted in the past can recover the other message by decoding the relay message and then inverting the linear operation" in which this will explore the achievable rate of the generated codes [6].

B. The Resulting Rates

These rates are directly taken from the paper. $R_1 < I(X_1, X_2; Y_3|X_3)$ (1)

$R_3 < I(X_2, X_3; Y_1|X_1)$ (2) and

$R_1 < I(X_1; Y_2|X_2, X_3)$ (3)

$R_3 < I(X_3; Y_2|X_1, X_2)$ (4)

$R_1 + R_3 < I(X_1, X_3; Y_3|X_2)$ (5)

for any $p(x_1), p(x_2), p(x_3)$ where 1,3 are source nodes and 2 is the relay node [6].

(1) and (2) are the cut-set bounds, (3)-(5) is the multiple-access region if the relay is to fully decode both sources [6]. So far this is the one-way relay model.

Now extending this to the two-way two-way relay channel will result in the following:

$R_1 < I(X_1, X_2, X_3; Y_4|X_4)$ (6)

$R_4 < I(X_2, X_3, X_4; Y_1|X_1)$ (7) and

$R_1 < I(X_1; Y_2|X_2, X_3, X_4)$ (8)

$R_4 < I(X_3, X_4; Y_2|X_1, X_2)$ (9)

$R_1 + R_4 < I(X_1, X_3, X_4; Y_2|X_2)$ (10) and

$R_1 < I(X_1, Y_2; Y_3|X_3, X_4)$ (11)

$R_4 < I(X_4; Y_3|X_1, X_2, X_3)$ (12)

$R_1 + R_4 < I(X_1, X_2, X_4; Y_3|X_4)$ (13)

where 1,4 are source nodes and 2,3 are the relay nodes.

(6) and (7) are the cut-set bounds, (8)-(10) is the multiple-access constraint.

C. The Idea

Xie and Ponniah proposed that to achieve (8) - (10) node 3 needs to decode before node 2 and also the reverse is also needed for (11)-(13). This deadlock problem is solved by adding an additional constraint on (6)-(13) that ensures some relay can decode at least one of the sources before the other [6].

D. Results

It is concluded that the rate is achievable and the deadlock problem was resolved by the additional constraint on (6)-(13), and here is the constraints [6]: $R_1 < I(X_1; Y_2|X_2, X_3)$ [6]

$R_4 < I(X_4; Y_3|X_2, X_3)$ [6]

$R_1 + R_4 < \max[I(X_1, X_4; Y_2|X_2, X_4), I(X_1, X_4; Y_3|X_2, X_3)]$ [6]
VII. REFERENCES