On the Analysis of a Transmission Line with Nonlinear Terminations using the Time Dependent BLT Equation

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Abstract

In this paper, the Baum-Liu-Tesche (BLT) equation is extended from the frequency domain into the time domain, so as to permit an analysis of a transmission line with nonlinear loads at each end of the line. To do this, we use both the voltage and current BLT equations in the time domain, together with the nonlinear v-i relationships for the termination impedances, to formulate a nonlinear BLT equation for the unknown reflection coefficients at each end of the line. This equation is solved by a time-marching procedure and this provides the load reflection coefficients for the nonlinear terminations. Once the reflection coefficients are determined, the load voltages and currents can be computed from the transient BLT equations. This computational procedure is illustrated with several examples.

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1. Introduction

In an earlier paper [1], a time-domain BLT equation for a lossy transmission line was derived from its frequency-domain counterpart. Based on an early paper by Baum for a general multiconductor transmission line network [2], the transient BLT solution was specialized to the case of a single transmission line section to illustrate this procedure. The time-domain BLT equation permits the determination of the currents and voltages at the terminating loads of the line directly in the time domain by evaluating several convolution integrals. Implicit in this solution are the assumptions that the termination loads at the line ends are linear, and that the losses in the line are modeled by the skin effect phenomena having a \( \sqrt{f} \) dependence of the per-unit-length (PUL) resistance of the line [3].

The solution in [1] differs from other transient solutions for lossy transmission lines in that analytical expressions for the transient voltage wave propagation function \( g(t,x) \), the surge characteristic impedance \( z_{c}(t) \) and admittance \( y_{c}(t) \), and the transient voltage reflection coefficients \( r_{1}(t) \) and \( r_{2}(t) \) at each end of the line are used in the solution. This provides a solution that is different from other commonly used analysis methods, such as the frequency-domain analysis coupled with a fast Fourier transform (FFT) [4], the distributed circuit model approach [5], and the finite-difference, time-domain (FDTD) solution of the telegraphers’ equations [6].

In the transient BLT solution of [1], the functions \( g(t,x) \), \( z_{c}(t) \) and \( y_{c}(t) \) depend only on the transmission line parameters, and as such, they can be evaluated from the inverse Laplace transforms of the frequency-domain parameters \( G(s,x) = e^{-\gamma s} \), \( Z_{c}(s) \) and \( Y_{c}(s) \). The transient reflection coefficients \( r_{1}(t) \) and \( r_{2}(t) \) depend on both the line parameters and the terminations, and for linear loads on the line, these functions are the inverse transforms of the frequency-domain reflection coefficients \( \rho_{1}(s) \) and \( \rho_{2}(s) \).

If all of the transient parameters are known, the solution for the load responses can be evaluated by simply computing the required convolution integrals by a direct integration process. As noted in [1], this solution is identical to that provided by a frequency-domain/FFT analysis up to a pre-determined time for which the transient solution becomes inaccurate. If the loads on the transmission line are nonlinear, however, it is impossible to determine the reflection coefficients \( r_{1}(t) \) and \( r_{2}(t) \) prior to obtaining a solution, because the values of these parameters depend on the solution. Thus, a modification of the transient BLT solution in [1] is required to permit the treatment of nonlinear loads on the line. This is the subject of the present paper.
2. Development of the Time-Domain BLT Equations for Nonlinear Loads

The problem to be discussed here is shown in Figure 1. It consists of a lossy transmission line of length $L$, with time dependent and/or nonlinear loads $Z_{L1}(t)$ and $Z_{L2}(t)$ at each end of the line. The line is excited by a lumped transient voltage and current source at location $x = x_s$. For this problem the transient load voltage and currents at each end ($v_1(t)$, $i_1(t)$) and ($v_2(t)$, $i_2(t)$) are required.

![Figure 1. A lossy transmission line with nonlinear loads at each end and excited by lumped transient voltage and current sources.](image)

To develop the nonlinear BLT solution, we will first review the transient BLT solution for the case of linear loads, and then extend it to consider load nonlinearities.

2.1 Review of the Transient BLT Equations for Linear Loads

Under the assumptions that there are no dielectric losses in the line and that the transmission line is described by a constant per-unit-length inductance $L'$, capacitance $C'$, and dc and resistance $R'$, along with a skin-loss term that is proportional to $\sqrt{s}$ , ref. [1] constructs the total per-unit-length impedance and admittance parameters of the line in the frequency domain as

$$Z' = sL' + R' + \xi \sqrt{s} \quad \text{and} \quad Y' = sC'. \quad (1)$$

For the special case of two circular conductors of radii $a$ and $b$, or for a coaxial line with conductor radii $a$ and $b$, the parameter $\xi$ is given by

$$\xi = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\sigma}} \left( \frac{1}{a} + \frac{1}{b} \right). \quad (2)$$

With these line parameters, the transmission line propagation constant is

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1 In this paper we use Laplace transform variable $s = \sigma + j\omega$ to denote the complex frequency.
\[ \gamma(s) = \sqrt{sL' + \xi \sqrt{s} + R'} sC' \]  
(3)

and the characteristic impedance of the line is expressed as

\[ Z_c(s) = \sqrt{\frac{(sL' + \xi \sqrt{s} + R')}{sC'}}. \]  
(4)

### 2.1.1 The Transient Propagation Function \( g(x,t) \)

For a low-loss line and sufficiently high frequencies, where \( \frac{\xi}{L' \sqrt{s}} \ll 1 \) and \( |R'/L'| \ll 1 \), ref.[1] approximates the propagation constant in Eq.(3) as

\[ \gamma(s) \approx \frac{s}{v} + \sqrt{s}a_1 + a_2 \]  
(5)

where

\[ v = \frac{1}{\sqrt{L'C'}} \quad a_1 = \frac{\xi}{2 \sqrt{L'}} \quad a_2 = \frac{R'}{2 \sqrt{L'}}. \]  
(6)

Under this low loss approximation, the propagation function of the line in the frequency domain is given as

\[ G(s; x) = e^{-\gamma(s)x} \approx e^{-\frac{s}{v}x} e^{-\sqrt{s}a_1 x} e^{-a_2 x}, \]  
(7)

and its inverse Laplace transform is shown to be

\[ g(t; x) = e^{-a_2 t} \frac{a_1 x}{2\sqrt{\pi} \left( t - x/v \right)^3} e^{\left( \frac{(a_1 x)^2}{4(t-x/v)} \right)} \quad \text{for } t > x/v. \]  
(8)

This function \( g(t; x) \) is the transient response for the line propagation, and it provides the distorted shape of an applied impulse function at \( x = 0 \), after it has propagated through a distance \( x \) along the line.

### 2.1.2 Surge Impedance and Admittance

For the same low loss assumptions used in Section 2.1.1, the frequency-domain characteristic impedance of Eq.(4) may be approximated as

\[ Z_c(s) \approx \hat{Z_c} \left( 1 + \frac{R'}{2sL'} + \frac{\xi}{2\sqrt{s}L'} \right). \]  
(9)
where
\[ \hat{Z}_c = \sqrt{\frac{L'}{C'}} \]  
(10)
is the high frequency limit of the characteristic impedance of the line.

The transient counterpart to \( Z_c(s) \) is the surge impedance \( z_c(t) \) and this is expressed in [1] as
\[
z_c(t) = \hat{Z}_c \delta(t) + \hat{Z}_c \frac{R'}{2L'} \Phi(t) + \hat{Z}_c \frac{\xi}{2L'} \frac{1}{\sqrt{\pi t}} \]  
(11)
where \( \Phi(t) \) is the unit step function and \( \delta(t) \) is the Dirac delta function.

A similar procedure can be used to transform the characteristic admittance of the line into the surge admittance \( y_c(t) \). The result is
\[
y_c(t) = \hat{Y}_c \delta(t) - \hat{Y}_c \frac{R'}{2L'} \Phi(t) - \hat{Y}_c \frac{\xi}{2L'} \frac{1}{\sqrt{\pi t}} \]  
(12)
where
\[
\hat{Y}_c = \hat{Z}_c^{-1} = \sqrt{\frac{C'}{L'}}. \]  
(13)

2.1.3 The Transient Reflection Coefficients

The transient reflection coefficient (one for each end of the line) depends both on the line properties and on the nature of the termination impedances. In the frequency domain the voltage reflection coefficient is
\[
\rho(s) = \frac{R_L - Z_c(s)}{R_L + Z_c(s)} = \frac{R_L - \hat{Z}_c \left( 1 + \frac{R'}{2sL'} + \frac{\xi}{2\sqrt{s}L'} \right)}{R_L + \hat{Z}_c \left( 1 + \frac{R'}{2sL'} + \frac{\xi}{2\sqrt{s}L'} \right)} \]  
(14)
where \( R_L \) denotes load attached to the end of the line. This load is assumed to be linear and only resistive. The inverse transform of this reflection coefficient is provided in [1] as
\[
r(t) = b_0 \delta(t) + b_1 \left( \frac{1}{\sqrt{\pi t}} - \xi e^{-\xi^2t} \text{erfc} \left( \xi \sqrt{t} \right) \right) - b_2 \left( \frac{1}{\sqrt{\pi t}} - \xi_2 e^{-\xi_2^2t} \text{erfc} \left( \xi_2 \sqrt{t} \right) \right) \]  
(15)
with the following parameters being defined in terms of the line and load constants:

\[ b_0 = \frac{R_L - \hat{Z}_c}{R_L + \hat{Z}_c} \]  
(16a)

\[ b_1 = \frac{R_L \hat{Z}_c}{2} \left( \frac{\xi^2 \hat{Z}_c - \xi \sqrt{\hat{Z}_c \left( \xi^2 \hat{Z}_c - 8R' L \left( R_L + \hat{Z}_c \right) \right) - 4R' L' \left( R_L + \hat{Z}_c \right) \right)}{(R_L + \hat{Z}_c)^2 L' \sqrt{\hat{Z}_c \left( \xi^2 \hat{Z}_c - 8R' L' \left( R_L + \hat{Z}_c \right) \right) \right)} \right) \]  
(16b)

\[ b_2 = \frac{R_L \hat{Z}_c}{2} \left( \frac{\xi^2 \hat{Z}_c + \xi \sqrt{\hat{Z}_c \left( \xi^2 \hat{Z}_c - 8R' L \left( R_L + \hat{Z}_c \right) \right) - 4R' L' \left( R_L + \hat{Z}_c \right) \right)}{(R_L + \hat{Z}_c)^2 L' \sqrt{\hat{Z}_c \left( \xi^2 \hat{Z}_c - 8R' L' \left( R_L + \hat{Z}_c \right) \right) \right)} \right) \]  
(16c)

\[ \zeta_1 = -\frac{1}{4(R_L + \hat{Z}_c)L'} \left[ \xi \hat{Z}_c - \sqrt{\hat{Z}_c \left( \xi^2 \hat{Z}_c - 8R' L' \left( R_L + \hat{Z}_c \right) \right) \right] \]  
(16d)

\[ \zeta_2 = -\frac{1}{4(R_L + \hat{Z}_c)L'} \left[ \xi \hat{Z}_c + \sqrt{\hat{Z}_c \left( \xi^2 \hat{Z}_c - 8R' L' \left( R_L + \hat{Z}_c \right) \right) \right] \]  
(16e)

### 2.1.4 The Transient BLT Equations for Linear Loads

Reference [1] develops the transient BLT equation for the voltages for linear loads as

\[
\begin{bmatrix}
    v_1(t) \\
    v_2(t)
\end{bmatrix}
= \begin{bmatrix}
    \delta(t) + r_1(t) \\
    0
\end{bmatrix} \otimes m(t, r_1, r_2) \otimes \begin{bmatrix}
    \delta(t) \\
    g(t, L) \otimes r_2(t)
\end{bmatrix} \otimes \begin{bmatrix}
    s_1(t) \\
    s_2(t)
\end{bmatrix},
\]

(17a)

where the transient source vector is written in terms of the voltage and current source waveforms as

\[
\begin{bmatrix}
    s_1(t) \\
    s_2(t)
\end{bmatrix} = \begin{bmatrix}
    -\frac{1}{2} \left( v_s(t) - z_c(t) \otimes i_s(t) \right) \otimes g(t, z_c) \\
    \frac{1}{2} \left( v_s(t) + z_c(t) i_s(0) \right) \otimes g(t, L - z_c)
\end{bmatrix}.
\]

(17b)

In these equations the symbol \( \otimes \) denotes the convolution operation. The transient functions \( g, r_1, r_2 \) and \( z_c \) used in Eqs.(17) are those previously discussed in Sections 2.1.1 – 2.1.3, and the function \( m(t) \) is determined from an expansion of the resonance matrix of the frequency-domain BLT equation, and it takes into account all of the multiple reflections on the line. It is given by the infinitely nested convolution product as

\[
m(t, r_1, r_2) = \delta(t) + m_o(t) \otimes \left\{ \delta(t) + m_o(t) \otimes \left[ \delta(t) + m_o(t) \otimes (\cdots) \right] \right\}
\]

(18a)
with

\[ m_t(t) \equiv g(t, L) \otimes g(t, L) \otimes r_1(t) \otimes r_2(t). \]  \hspace{1cm} (18b)

In Eq. (18a) the arguments of the transient function \( m \) are listed as \( (t, r_1, r_2) \) to draw attention to the fact that this function depends explicitly on the load reflection coefficients. In any practical calculation, the convolutions in Eq. (18a) must be terminated at some point, and this introduces late-time errors in any transient solution.

In examining the frequency-domain BLT equation for the load currents in [1], the transient counterpart to this equation can be obtained by inspection as

\[

\begin{bmatrix}
    i_1(t) \\
    i_2(t)
\end{bmatrix} = \begin{bmatrix}
    \delta(t) - r_t(t) & 0 \\
    0 & \delta(t) - r_t(t)
\end{bmatrix} \otimes \begin{bmatrix}
    y_1(t) \otimes m(t, r_1, r_2) \otimes
    \begin{bmatrix}
        \delta(t) & g(t, L) \otimes r_1(t) \\
        g(t, L) \otimes r_2(t) & \delta(t)
    \end{bmatrix} \otimes
    \begin{bmatrix}
        s_1(t) \\
        s_2(t)
    \end{bmatrix}
\end{bmatrix}
\]  \hspace{1cm} (19)

For linear loads, the solutions to Eqs. (17) and (19) are relatively straightforward, as each of the convolution operations can be performed independently from the others. Thus, a time-stepping procedure is not required. The various functions and convolutions are evaluated and used in Eqs. (17) and (19) to determine the time-dependent current and voltage vectors. It is important to note that these equations may be evaluated independently of each other – a benefit that does not occur in the nonlinear load model that will be developed in the next section.

### 2.2 The Transient BLT Equations for Nonlinear Loads

The entire computational procedure for determining the load responses must be revised if the loads are nonlinear, due to the fact that the reflection coefficients are not known a priori. One approach described by Djordjevic [7] employs the method developed by Liu and Tesche [8] for using a time-domain Norton or Thevenin equivalent circuit for the linear portion of the transmission line, and then using this in a nonlinear circuit analysis at the ports where nonlinearities are located. While the transient BLT equations (17) and (19) could be used in this manner, it is instructive to show how to include the nonlinearities directly in a BLT solution.

To treat the nonlinear transmission line load problem with the BLT equation, we will first assume that the nonlinearities at both loads are described by a simple \( v-i \) relationship of the form

\[ v(t) = F[i(t)]. \]  \hspace{1cm} (20)

This instantaneous, although nonlinear, relationship between \( v \) and \( i \) precludes the possibility of having energy storage mechanisms present in the load. Denoting by \( F_1(\cdot) \) and \( F_2(\cdot) \) the nonlinear load operators for loads 1 and 2, respectively, we can develop the following nonlinear matrix load operator for the line:
\[
\begin{bmatrix}
  v_1(t) \\
  v_2(t)
\end{bmatrix} = \begin{bmatrix}
  F_1(\cdot) & 0 \\
  0 & F_2(\cdot)
\end{bmatrix}
\begin{bmatrix}
  i_1(t) \\
  i_2(t)
\end{bmatrix}
\]

By substituting Eqs. (17) and (19) into Eq. (21), we form the nonlinear BLT equation for the unknown transient reflection coefficients \( r_1(t) \) and \( r_2(t) \). This equation is given as

\[
\begin{bmatrix}
  \delta(t) + r_1(t) \\
  0
\end{bmatrix} \otimes m(t, r_1, r_2) \otimes \begin{bmatrix}
  \delta(t) & g(t, L) \otimes r_2(t) \\
  g(t, L) \otimes r_1(t) & \delta(t)
\end{bmatrix} \otimes \begin{bmatrix}
  s_1(t) \\
  s_2(t)
\end{bmatrix}
= \mathcal{F} \begin{bmatrix}
  \delta(t) - r_1(t) \\
  0
\end{bmatrix} \otimes y_c(t) \otimes m(t, r_1, r_2) \otimes \begin{bmatrix}
  \delta(t) & g(t, L) \otimes r_2(t) \\
  g(t, L) \otimes r_1(t) & \delta(t)
\end{bmatrix} \otimes \begin{bmatrix}
  s_1(t) \\
  s_2(t)
\end{bmatrix}
\]

To solve this equation for the transient reflection coefficients we note that \( r_1(t) \) and \( r_2(t) \) occur in several places. With the exception of the two diagonal matrices containing elements \((\delta(t) + r_1(t)), (\delta(t) + r_2(t)), (\delta(t) - r_1(t)) \) and \((\delta(t) - r_2(t))\), all of the other occurrences of the reflection coefficients are in convolutions with one or more transient propagation functions \( g(t,x) \). This implies that at a particular time \( t \) in the solution, only the reflection coefficients in these diagonal matrices are unknown, as the convolution operations provide a time shift and thereby use the reflection coefficients from an earlier time, which are presumably known. Thus, Eq. (22) amounts to a \( 2 \times 2 \) nonlinear matrix BLT equation that can be time-stepped along with a nonlinear matrix equation being solved at each time step.

To better illustrate this solution, Eq. (22) can be split into two simultaneous nonlinear equations for the reflection coefficients. These are

\[
\begin{align*}
(\delta(t) + r_1(t)) \otimes m(t, r_1, r_2) \otimes [s_1(t) + r_2(t) \otimes g(t, L) \otimes s_2(t)] \\
= F_1 \left\{ (\delta(t) - r_1(t)) \otimes m(t, r_1, r_2) \otimes [y_c(t) \otimes s_1(t) + r_2(t) \otimes y_c(t) \otimes g(t, L) \otimes s_2(t)] \right\}
\end{align*}
\]

for load #1, and

\[
\begin{align*}
(\delta(t) + r_2(t)) \otimes m(t, r_1, r_2) \otimes [r_1(t) \otimes g(t, L) \otimes s_1(t) + s_2(t)] \\
= F_2 \left\{ (\delta(t) - r_2(t)) \otimes m(t, r_1, r_2) \otimes [r_1(t) \otimes y_c(t) \otimes g(t, L) \otimes s_1(t) + y_c(t) \otimes s_2(t)] \right\}
\end{align*}
\]

for load #2. By defining the following convolution functions having the units of voltage at loads #1 and #2 as

\[
\Theta_1^{inc} (t) = m(t, r_1, r_2) \otimes [s_1(t) + r_2(t) \otimes g(t, L) \otimes s_2(t)]
\]

for load #1, and

\[
\Theta_2^{inc} (t) = m(t, r_1, r_2) \otimes [r_1(t) \otimes g(t, L) \otimes s_1(t) + y_c(t) \otimes s_2(t)]
\]

for load #2.
we can write the nonlinear BLT equations in (23) as

$$\left( \delta(t) + r_i(t) \right) \otimes \Theta_{inc}^i(t) = F_i \left\{ \left( \delta(t) - r_i(t) \right) \otimes \Upsilon_{inc}^i(t) \right\} \quad \text{(for load #1)} \quad (25a)$$

and

$$\left( \delta(t) + r_i(t) \right) \otimes \Theta_{inc}^i(t) = F_i \left\{ \left( \delta(t) - r_i(t) \right) \otimes \Upsilon_{inc}^i(t) \right\} \quad \text{(for load #2)} \quad (25b)$$

The functions $\Theta_{inc}$ and $\Upsilon_{inc}$ correspond to incident voltage and current waves that are incident on the loads 1 and 2 at time $t$. Because of the various convolutions that enter into the definitions of $\Theta_{inc}$ and $\Upsilon_{inc}$, they depend only on past values of the reflection coefficients $r_1$ and $r_2$, and thus, at a time $t$, these functions are known.

To obtain a solution for the reflection coefficient $r_1$, Eq.(25a) can be written in a discrete form for times $t_i \ (i = 0, 1, 2 \cdots)$, with $\Delta$ being the time step $(t_i-t_0)$, as

$$\sum_{k=0}^{i} \left( \delta_k + r_i \Delta \right) \Theta_1^{inc} (t_{i-k}) = F_i \left\{ \sum_{k=0}^{i} \left( \delta_k - r_i \Delta \right) \Upsilon_1^{inc} (t_{i-k}) \right\} \quad i = 0, 1, 2, \cdots. \quad (26)$$

where $\delta_k$ is the kronecker delta function: $\delta_k = 1$ for $k = 0$, and $= 0$, otherwise. At the start of the calculation where $i = 0$ and the time is $t_0$, all of the reflection coefficients are zero and Eq.(26) is given simply as

$$(1 + r_{i0} \Delta) \Theta_1^{inc} (t_0) = F_i \left\{ (1 - r_{i0} \Delta) \Upsilon_1^{inc} (t_0) \right\} \quad i = 0. \quad (27)$$

With the functions $\Theta_{inc}$ and $\Upsilon_{inc}$ evaluated at $t_0$, this equation amounts to a nonlinear equation to be solved for the load #1 reflection coefficient $r_{i0}$ in the initial time step. This solution is obtained numerically by determination the value of $r_{i0}$ that satisfies Eq.(27).

In the next time step, where $i=1$, Eq.(26) is becomes

$$(1 + r_{i0} \Delta) \Theta_1^{inc} (t_1) + r_i \Delta \Theta_1^{inc} (t_0) = F_i \left\{ (1 - r_{i0} \Delta) \Upsilon_1^{inc} (t_1) - r_i \Delta \Upsilon_1^{inc} (t_0) \right\} \quad i = 1. \quad (28)$$
The reflection coefficient \( r_{10} \) is known from the previous time step, but now the reflection coefficient \( r_{11} \) is still unknown. This coefficient can be determined as done previously he previous time step by numerically solving for the value of \( r_{11} \) that satisfies Eq.(28).

For a later time step where \( i > 1 \), the nonlinear equation required for solving for the unknown coefficient \( r_{1i} \) is

\[
\sum_{k=0}^{i-1} (\delta_k + r_{ik} \Delta) \Theta_{i}^{inc} (t_{i-k}) + r_i \Delta \Theta_{i}^{inc} (t_0) = F_i \left\{ \sum_{k=0}^{i-1} (\delta_k - r_{ik} \Delta) \gamma_{i}^{inc} (t_{i-k}) - r_i \Delta \gamma_{i}^{inc} (t_0) \right\} \quad i > 1
\]

This expression is simply Eq.(25) expanded to show explicitly the unknown coefficient \( r_{1i} \) that is to be determined in this time step.

Ultimately, the time-marching solution of Eq.(29) will be in error, because the functions \( \Theta^{inc} \) and \( \gamma^{inc} \) depend on prior values of the reflection coefficient \( r_2 \) at load #2 and this parameter is not being updated in the solution. Consequently in each time step we also need to time-march Eq.(25b) to determine \( r_2 \). Expanding this equation in the same manner as Eq.(29) provides the second general equation for \( r_2 \):

\[
\sum_{k=0}^{i-1} (\delta_k + r_{2k} \Delta) \Theta_{i}^{inc} (t_{i-k}) + r_2 \Delta \Theta_{i}^{inc} (t_0) = F_i \left\{ \sum_{k=0}^{i-1} (\delta_k - r_{2k} \Delta) \gamma_{i}^{inc} (t_{i-k}) - r_2 \Delta \gamma_{i}^{inc} (t_0) \right\} \quad i > 1
\]

with equations to similar (27) and (28) being used for the cases of \( i = 0 \) and 1.
3. Numerical Illustrations

To validate both the formulation and numerical implementation of the time marching procedure for Eqs.(29) and (30) we consider the numerical example presented in ref.[1] for a coaxial transmission line of length $L = 1$ m and single voltage source located at $x_s = 0.2$ m from the left end of the line (see Figure 1). The coaxial line has inner and outer conductor radii of $a = 2.5$ mm and $b = 9.345$ mm, respectively, with a thickness of the outer conductor $tk = 0.6$ mm. The cable is filled with a lossless dielectric with $\varepsilon_{rel} = 2.5$, and for these parameters, the high frequency characteristic impedance of the line is $50 \, \Omega$. For this example, the termination impedances are assumed to be linear and completely resistive, with values $Z_{L1} = R_{L1} = 100 \, \Omega$ and $Z_{L2} = R_{L2} = 10 \, \Omega$.

The excitation voltage waveform is taken to be a fast transient having a rise time of about $100$ ps and a fall time of roughly $4$ ns. As developed by Giri [9], this pulse is modeled by the expression

$$V_s(t) = V_p \left(1 + \Gamma\right) e^{-\left(t/t_r\right)} \left[0.5 \text{ erfc}\left(-\frac{t}{\sqrt{2} \, t_r}\right) \Phi(-t-t_s) + \left[1 - 0.5 \text{ erfc}\left(\frac{t}{\sqrt{2} \, t_r}\right)\right] \Phi(t-t_s)\right] \tag{31}$$

where $\text{erfc} \,(\cdot)$ is the complementary error function and $\Phi(\cdot)$ is the unit step function. The following waveform parameters are used for this example: $V_p = 10$ (kV), $\Gamma = 0.024$, $t_r = 100$ (ps), $\tau_f = 4$ (ns), and $\tau_s = 0.2$ (ns). Figure 2 plots the voltage waveform for these parameters.

3.1 Calculation with a Linear Load

Once the expressions in Eqs.(29) and (30) are used for determining the transient reflection coefficients $r_1$ and $r_2$, Eqs.(17) and (19) can be evaluated to provide the transient load voltages and currents for the line. For the coaxial line is made of copper with conductivity $\sigma = 5.76 \times 10^7$ S/m, Figure 3 plots the resulting transient voltages and currents at each of the loads. The solid curves represent the solution using the BLT equation time marching procedure, and the overlay dots denote the responses obtained using a conventional frequency-domain analysis with Schelkunoff’s analytical expression for the loss in the coax line [10], together with a FFT. The agreement between these two solutions is excellent.
Figure 3. Linear and log plots of the load voltages and currents for the case of a highly conducting (copper) coaxial line. (Solid lines are for the time-marching solution and the dots represent the frequency-domain/FFT solution.)

It is also possible to validate the numerical implementation of the time marching algorithm by checking the computed reflection coefficients with those obtained from the analytical solution described in ref. [1]. Figure 4 shows this comparison for both loads. The agreement is excellent, except for a few times, where the computed reflection coefficients appear to have a some spikes. The cause for these few differences is still under examination, but it is interesting to point out that as the time increment of the numerical solution $\Delta t$ increases, the heights of these spikes increase and the early-time comparison of the computed and analytical results begin to deviate. These spikes may arise from poor convergence of the iteration solution for the reflection coefficients.

Figure 4. Comparison of the transient reflection coefficients $r_1$ and $r_2$ for loads #1 and #2, as determined from the time marching solution (solid lines) and from the analytical expression of ref. [1] (dots).
3.2 Calculation with a Nonlinear Load

The benefit of the time-marching BLT solution is that it is able to handle time-varying and nonlinear loads. To illustrate how this is done, consider the transmission line geometry just analyzed for the linear load case, but now with a nonlinear resistor located at load #1. For this load, we will assume that it is a linear load with a resistance of 100 $\Omega$ for voltages up to 2 kV. For higher voltages applied across the load, the resistance switches to 1 $\Omega$. The behavior of this load is similar to a voltage clamping device and it is modeled by the following function:

$$ v(t) = F(i(t)) = R_L i(t) \quad \text{for} \quad |i(t)| \leq V_b $$

$$ = V_b + \left( i(t) - \frac{V_b}{R_L} \right) R_1 \quad \text{for} \quad i(t) > V_b $$

$$ = -V_b + \left( i(t) + \frac{V_b}{R_L} \right) R_1 \quad \text{for} \quad i(t) < -V_b $$

(32)

with $V_b = 2$ kV, $R_L = 100$ $\Omega$ and $R_1 = 1$ $\Omega$. A plot of this function is shown in Figure 5 as a function of the applied current to the load.

![Figure 5. Plot of a simple nonlinear v-i curve for load $R_{L1}$.](image)

The solid curves in Figure 6 represent the transient load voltages and currents for the nonlinear element located at load #1. Shown by the dotted lines are the corresponding linear load responses, which were presented previously in Figure 3. We note in Figure 6 that the load #1 voltage is clipped at about -2 kV, and that as a consequence, the current flowing through this load is about a factor of 2 larger than that for the linear load.

It is also apparent from Figure 6 that the operation of the nonlinear element at load #1 affects the response at load #2. For the nonlinear element present, we note that there is an increase in the amplitudes of both the voltage and current across $R_{L2}$ and the pulse duration is longer. Thus, there is more instantaneous power delivered to $R_{L2}$, with an increased chance of failure in this element. This example illustrates the well-known effect of a protective device (a voltage clamp in this case) increasing the stress on other components in the system.
Had the element at load #2 been nonlinear as well, the additional nonlinearity could have been incorporated in the time-marching solutions of Eqs.(29) and (30) with no difficulty.

![Load Voltages and Currents](image)

**Figure 6.** Plots of the load voltages and currents for the case of the line with a nonlinear resistance at load #1. (Solid lines are for the nonlinear response, and the dotted lines represent the linear solution.)

The time-marching BLT solution will also work with bi-polar waveforms, such as the one shown in Figure 7. For this case a 4 ns time window contains 2 cycles of a 500 MHz sinusoidal signal, and this is applied as the excitation source to the transmission line. Figure 8 presents the resulting linear and nonlinear responses at load #1. The voltage clipping at the nonlinear element at load #1 is clearly evident. The responses for load #2 are similar, but are not plotted here.

![Sinusoidal Waveform](image)

**Figure 7.** Plots of a 2-cycle sine wave voltage source with frequency $f = 500$ MHz.
4. Conclusion

This paper has described the use of the transmission line BLT equation for computing the transient voltage and current responses for nonlinear loads attached to the line.

Normally, the BLT equation for a single transmission line, or for an extended network of such lines, is formulated in the frequency domain, and it is required that the line termination impedances be linear. As discussed in [1], for linear loads it is possible to convert the frequency-domain BLT equation into a time-domain BLT equation, in which convolution operations of various transient functions replace multiplications of the spectral domain counterparts of the transient functions. These transient functions are the line propagation impulse response, the surge impedance and admittance of the line, and the reflection coefficients at each of the loads. In [1], analytical expressions for each of these transient functions are provided for the case of a coaxial cable, and illustrations of selected transient responses are provided.

In this paper, the transient BLT formalism is extended to permit the analysis of a transmission line with a nonlinear load at each end of the line. To do this, it is necessary to use the transient BLT equations for both the load voltages and the load currents, and relate these to each other through the nonlinear v-i relationships of the terminations. This results in a nonlinear BLT matrix equation (Eq.(22)) that must be stepped along in time to effect a solution.

Several numerical examples have been presented in this paper to illustrate the use of this technique, and the results appear reasonable. While it is clear that this technique works well for simple resistive nonlinearities on a single transmission line, additional work needs to be undertaken in this area. Interesting issues for further study include:
• develop an extension of the present technique for nonlinear load analysis to a multiconductor line and to a network of such lines,
• extend the BLT analysis to include energy storage elements, both linear and nonlinear, first for a single transmission line and later, for a network,
• develop the nonlinear BLT analysis for a distributed EM field excitation,
• investigate the effects of having dielectric loss in the cables, which provide an additional per-unit-length conductance in the solution, and
• extend the BLT solution to transmission systems (such as a strip-line) that are not strictly described by the $\sqrt{f}$ functional dependence of the line loss.

5. References